Variational formulation of an ideal fluid and fluid Maxwell equations

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Outline

Part I. Variational formulation

There are two ways in description of fluid flows:
Lagrangian description and Eulerian description.

- Variational formualtions and Lagrangian functionals are reconsidered from the viewpoint of guage theory
  (expressed in a form independent of any particular coordinate system).
- Transformation between the two spaces are examined, and its uniqueness is considered.
- Importance of vorticity equation is emphasized in this regard.

Part II. Fluid Maxwell Equations

There is analogy between the equations of
Fluid mechanics and Electromagnetism.

- Defining two vector fields analogous to the electric and magnetic fields, fluid Maxwell equations are proposed.
- Vorticity corresponds to the magnetic field.
- Equation of sound wave is derived. Hence, the sound wave is analogous to the Electromagnetic wave.
- Forces acting on a test particle moving in a flow field are expressed with the same form as of the electromagnetism.

Part I. Variational formulation

1. Definitions

(a) Two descriptions

- **Lagrangian** *particle* coordinates:

$$\begin{aligned} \boldsymbol{a} &= (a^1, a^2, a^3) = (a, b, c), \qquad \text{time}: \quad \tau = t = a^0, \\ & a^\mu = (\tau, a^1, a^2, a^3), \qquad \mu = 0, 1, 2, 3 \\ & \mathrm{d}^3\boldsymbol{a} = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c \end{aligned}$$

- **Eulerian** *physical space* coordinates:

$$\begin{aligned} \boldsymbol{x} = (x^1, x^2, x^3) &= (x, y, z), \qquad \text{time}: \quad t \\ \text{volume element}: \quad \mathrm{d}^3\boldsymbol{x} &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \end{aligned}$$

Each particle is identified by $\boldsymbol{a} = (a, b, c)$. Its position is expressed with

$$\boldsymbol{X} = \boldsymbol{X}(\tau, \boldsymbol{a}) = (X^1, X^2, X^3) = (X, Y, Z), \qquad \boldsymbol{a} = \boldsymbol{X}(0, \boldsymbol{a})$$

Mass density ρ **:**

$$dm = \mathrm{d}^3\boldsymbol{a} = \rho(t, \boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x},$$

From $dm = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \rho\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$,

$$\rho = \frac{1}{J}, \qquad J = \frac{\partial(X^k)}{\partial(a^l)} = \frac{\partial(X^1, X^2, X^3)}{\partial(a^1, a^2, a^3)} = \frac{\partial(X, Y, Z)}{\partial(a, b, c)} \qquad \text{(Jacobian)}$$

(b) Ideal Fluid

The mass dm of a fluid particle is **invariant** during its motion:

$$\partial_\tau(\mathrm{d}m) = 0\,,$$

$$\partial_\tau \equiv \partial/\partial\tau, \qquad \text{for fixed } \boldsymbol{a}.$$

There is no dissipation of kinetic energy. Therefore,

$\Rightarrow$ no heat generation, $\Rightarrow$ no entropy change:

$$\partial_\tau\, s = 0\,, \qquad \text{(Definition of an ideal fluid).}$$

$$\Rightarrow \quad s = s(\boldsymbol{a}) = s(a, b, c).$$

(c) Thermodynamic relations: Setting as $\delta s = 0$ (vanishing variation),

$$\delta\epsilon = (\delta\epsilon)_s = \frac{p}{\rho^2}\,\delta\rho, \qquad \delta h = (\delta h)_s = \frac{1}{\rho}\delta p,$$

$(\,\cdot\,)_s$ denotes variation with s fixed.

p : *pressure*

ϵ, $h\,(= \epsilon + p/\rho)$: *internal energy*, *enthalpy*, (per unit mass).

For small changes of $\delta\rho$ (density) and δs (entropy):

$$\delta\epsilon = (p/\rho^2)\delta\rho + T\delta s\,, \qquad \delta h = (1/\rho)\delta p + T\delta s.$$

2. Lagrangian description

(*a*) Lagrangian functional $\mathcal{L}$ is defined by

$$\mathcal{L} = \int_{M_a} \Lambda(X^k_\mu, X^k)\, \mathrm{d}^3\boldsymbol{a}$$

$$\Lambda(X^k_\mu, X^k) = \tfrac{1}{2}\, X^k_\tau\, X^k_\tau - \epsilon(\rho, s)\,, \qquad (1)$$

(kinetic energy) $-$ (internal energy)

X^k_τ : k-component velocity, $\qquad \epsilon$: internal energy.

$$\text{Action :}\quad I = \int_{\tau_1}^{\tau_2} \mathcal{L}\, \mathrm{d}\tau = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \int_{M_a} \Lambda(X^k_\mu, X^k)\, \mathrm{d}^3\boldsymbol{a}, \qquad (2)$$

Lagrangian density :

$$\Lambda = \Lambda(X^k_\mu, X^k) = \tfrac{1}{2}\, X^k_0\, X^k_0 - \epsilon(X^k_l, X^k), \qquad (3)$$

$$X^k_\mu = \partial X^k / \partial a^\mu .$$

Variation :

$$\begin{aligned} \delta\Lambda &= \frac{\partial \Lambda}{\partial X^k}\delta X^k + \frac{\partial \Lambda}{\partial X^k_\mu}\delta X^k_\mu \\ &= \left[\frac{\partial \Lambda}{\partial X^k} - \frac{\partial}{\partial a^\mu}\left(\frac{\partial \Lambda}{\partial X^k_\mu}\right)\right] \delta X^k + \frac{\partial}{\partial a^\mu}\left(\frac{\partial \Lambda}{\partial X^k_\mu}\delta X^k\right) \end{aligned}$$

$$\delta I = \int_{\tau_1}^{\tau_2} \int_{M_a} \delta\Lambda\ \ \mathrm{d}^4 a = 0, \qquad \mathrm{d}^4 a = \mathrm{d}\tau\, \mathrm{d}^3\boldsymbol{a}.$$

(b) Euler-Lagrange equation

$$\frac{\partial}{\partial a^\mu}\left(\frac{\partial \Lambda}{\partial X^k_\mu}\right) - \frac{\partial \Lambda}{\partial X^k} = 0, \qquad k = 1,2,3; \quad \mu = 0,\cdots,3. \tag{4}$$

Noether's theorem :

$$\frac{\partial}{\partial a^\nu} T^\nu_\mu = 0$$

$$T^\nu_\mu \equiv X^k_\mu \left(\frac{\partial \Lambda}{\partial X^k_\nu}\right) - \Lambda\, \delta^\nu_\mu \qquad \text{(Energy-momentum tensor)}$$

$$\frac{\partial}{\partial a^\nu} T^\nu_\mu = X^k_\mu \left[\partial_\nu \left(\frac{\partial \Lambda}{\partial X^k_\nu}\right) - \frac{\partial \Lambda}{\partial X^k}\right] = 0\,.$$

Eckart (1960), Kambe (2008)

(i) Energy equation ($\mu = 0$):

$$\partial_\tau H \;+\; \partial_a\Big[p\frac{\partial(X,Y,Z)}{\partial(\tau,b,c)}\Big] + \partial_b\Big[p\frac{\partial(X,Y,Z)}{\partial(a,\tau,c)}\Big] + \partial_c\Big[p\frac{\partial(X,Y,Z)}{\partial(a,b,\tau)}\Big] = 0. \tag{5}$$

where $H = \frac{1}{2}\, v^2 + \epsilon$.

(ii) Equation of motion ($\mu \neq 0$):

$$\partial_\tau V_\alpha + \partial_\alpha F = 0, \qquad\qquad (\text{for } \alpha = a, b, c), \tag{6}$$

$$V_\alpha \;\equiv\; X_\alpha X_\tau + Y_\alpha Y_\tau + Z_\alpha Z_\tau, \qquad F = -\tfrac{1}{2}\, v^2 + h.$$

$(X_\tau,\, Y_\tau,\, Z_\tau)$: $\boldsymbol{x}$-space velocity

V_α : Velocity transformed to the $\boldsymbol{a}$-space.

From (6), the equation for the acceleration $\mathcal{A}_\alpha$ is

$$\mathcal{A}_\alpha(\equiv X_\alpha X_{\tau\tau} + Y_\alpha Y_{\tau\tau} + Z_\alpha Z_{\tau\tau}) = -\partial_\alpha h, \qquad (\alpha = a, b, c) \tag{7}$$

This is the **Lagrange's form of the equation of motion**.

Multiplying $\partial\alpha/\partial x$ and summing, we obtain

$$X_{\tau\tau} = -\frac{1}{\rho}\,\partial_x\, p, \qquad \partial_x p = \frac{\partial \alpha}{\partial x}\,\frac{\partial p}{\partial \alpha}. \tag{8}$$

Euler's equation of motion.

3. Transformation between (a, b, c) and (X, Y, Z)

- ***a*-space acceleration**,

$$\mathcal{A}_\alpha = \underline{X_\alpha X_{\tau\tau} + Y_\alpha Y_{\tau\tau} + Z_\alpha Z_{\tau\tau}} = \boldsymbol{e}_\alpha \cdot \mathcal{A},$$

Middle side is a form of an **inner product** of $\mathcal{A}$ and $\boldsymbol{e}_\alpha$ $(\alpha = a, b, c)$:

$$\begin{aligned} \boldsymbol{x}\text{-space acceleration} \quad &: \quad \mathcal{A} \equiv (X_{\tau\tau},\, Y_{\tau\tau},\, Z_{\tau\tau}), \\ \text{Direction vector of the } \alpha\text{-axis} \quad &: \quad \boldsymbol{e}_\alpha \equiv (X_\alpha,\, Y_\alpha,\, Z_\alpha). \end{aligned}$$

The inner product is **invariant** with respect to **rotation** of the frame axes (x, y, z) in the $\boldsymbol{x}$-space.

- ***a*-space velocity**:

$$V_\alpha = X_\alpha X_\tau + Y_\alpha Y_\tau + Z_\alpha Z_\tau = \boldsymbol{e}_\alpha \cdot V,$$

is also an inner product, where two vectors are $\boldsymbol{e}_\alpha$ and

$$\boldsymbol{x}\text{-space velocity} \quad : \quad V \equiv (X_\tau,\, Y_\tau,\, Z_\tau).$$

- **Transformation** between $(\Delta a, \Delta b, \Delta c)$ and $(\Delta X, \Delta Y, \Delta Z)$ is given by

$$\begin{aligned} \Delta X &= X_a \,\Delta a + X_b \,\Delta b + X_c \,\Delta c, \\ \Delta Y &= Y_a \,\Delta a + Y_b \,\Delta b + Y_c \,\Delta c, \\ \Delta Z &= Z_a \,\Delta a + Z_b \,\Delta b + Z_c \,\Delta c. \end{aligned}$$

There are **nine** undetermined coefficients: $X_a,\, X_b,\, X_c,\, \cdots,$ *etc.*.
But we have **six** relations of $\mathcal{A}_\alpha$ and V_α, **not sufficient** .

Additional transformation of **vorticity** must be considered, in order to fix this freedom for uniqueness. Kambe (2008).

4. Improved Lagrangians

We have three invariances:

$$\partial_\tau(\mathrm{d}^3\boldsymbol{a}) = 0, \qquad \partial_\tau s = 0, \qquad \partial_\tau\, \boldsymbol{\Omega}_a = 0.$$

Last one $\partial_\tau\, \boldsymbol{\Omega}_a = 0$ is obtained from *invariance* of $\mathcal{L}$ with respect to particle exchange, where

$$\begin{aligned} \boldsymbol{\Omega}_a &= \nabla_a \times \boldsymbol{V}_a &&: \quad \text{vorticity in the } \boldsymbol{a}\text{-space}, \\ \boldsymbol{V}_a &= (V_a, V_b, V_c) &&: \quad \text{velocity in the } \boldsymbol{a}\text{-space}. \end{aligned}$$

We can define the following improved Lagrangian (Kambe 2008):

$$\begin{aligned} \mathcal{L}^* &= \mathcal{L} - \partial_\tau \int \phi\, \mathrm{d}^3\boldsymbol{a} - \partial_\tau \int s\,\psi\, \mathrm{d}^3\boldsymbol{a} - \partial_\tau \int_M \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} \\ &= \mathcal{L} - \int_M \partial_\tau \phi\, \mathrm{d}^3\boldsymbol{a} - \int_M s\, \partial_\tau \psi\, \mathrm{d}^3\boldsymbol{a} - \int_M \langle \partial_\tau \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}. \end{aligned}$$

The action integral is unchanged by added Lagrangians:

$$I = \int_{\tau_1}^{\tau_2} \mathcal{L}^*\, \mathrm{d}\tau = \int \mathcal{L}\, \mathrm{d}\tau.$$

- The Euler-Lagrange equation is unchanged.
- However, these play non-trivial role in the Eulerian space.
- Particle permutation causes no change,
  since material particles are assumed to be uniform.
- Hence, there is arbitrariness in the definition of (a, b, c).
- The last term $\int \langle \partial_\tau \boldsymbol{A}_a, \boldsymbol{\Omega}_a \rangle \mathrm{d}^3\boldsymbol{a}$ is analogous to the Chern-Simons term in the Gauge Theory (where $\partial_\tau \boldsymbol{A}_a$ is assumed to be a function of $\boldsymbol{a}$ only).

5. Eulerian description

$(\boldsymbol{a})$ **Independent variables** are (t, x, y, z).

$$
\begin{array}{rll}
\text{Time derivative } \partial_\tau & \text{is replaced by} & \mathrm{D}_t \equiv \partial_t + v^k \partial_k = \partial_t + \boldsymbol{v} \cdot \nabla\,, \\
\text{Mass element } \mathrm{d}^3\boldsymbol{a} & \text{is replaced by} & \rho\, \mathrm{d}^3\boldsymbol{x}, \\
\text{Velocity } \partial_\tau \boldsymbol{X} & \text{is replaced by} & \boldsymbol{v} = (u, v, w) = \mathrm{D}_t \boldsymbol{x}.
\end{array}
$$

$(\boldsymbol{b})$ **Lagrangian in the Eulerian space**

$$
\mathcal{L}^* = \mathcal{L} - \int_M \mathrm{D}_t\phi\ \rho\, \mathrm{d}^3\boldsymbol{x} - \int_M s\, \mathrm{D}_t\psi\ \rho\, \mathrm{d}^3\boldsymbol{x} - \partial_\tau \int_M \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a}.
$$

The last integral can be given another equivalent forms:

$$
-\partial_\tau \int_M \langle \boldsymbol{A}_a,\, \boldsymbol{\Omega}_a \rangle\, \mathrm{d}^3\boldsymbol{a} = -\int_M \langle \overline{\boldsymbol{L}}_{\partial_\tau} \boldsymbol{A},\, \boldsymbol{\omega}^* \rangle\ \ \mathrm{d}^3\boldsymbol{x} \quad = \int_M \langle \boldsymbol{A},\, \overline{\boldsymbol{L}}_{\partial_\tau} \boldsymbol{\omega}^* \rangle\, \mathrm{d}^3\boldsymbol{x} + \mathrm{Int}_S
$$

The derivative ∂_τ is equivalent to the **Lie derivative** $\overline{\boldsymbol{L}}$, which is represented as follows:

$$
\begin{array}{rcl}
\text{Scalar } \Phi & : & \partial_\tau \Phi = \overline{\boldsymbol{L}}_{\partial_\tau} \Phi = \mathrm{D}_t \Phi = \partial_t \Phi + \boldsymbol{v} \cdot \nabla \Phi, \\
 & & \qquad \mathrm{D}_t a^k = 0\,, \\
\text{Cotangent vector } A_i & : & \partial_\tau A_k = \overline{\boldsymbol{L}}_{\partial_\tau} A_k = \partial_t A_k + (\boldsymbol{v} \cdot \nabla) A_k + A_i \partial_k v^i \\
 & & \qquad \Rightarrow \partial_t \boldsymbol{A} + \nabla(\boldsymbol{v} \cdot \boldsymbol{A}) - \boldsymbol{v} \times (\nabla \times \boldsymbol{A})\,, \\
\text{Axial vector } \boldsymbol{\omega}^* & : & \partial_\tau \boldsymbol{\omega}^* = \overline{\boldsymbol{L}}_{\partial_\tau} \boldsymbol{\omega}^* = \partial_t \boldsymbol{\omega}^* + \nabla \times (\boldsymbol{\omega}^* \times \boldsymbol{v}) + \boldsymbol{v}(\nabla \cdot \boldsymbol{\omega}^*).
\end{array}
$$

The action integral is represented as

$$I = \int_{\tau_1}^{\tau_2} \mathcal{L}^* \, \mathrm{d}\tau = \int \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}) \, \mathrm{d}^4 x, \qquad \mathrm{d}^4 x = \mathrm{d}t \, \mathrm{d}^3\boldsymbol{x}.$$

where the **Lagrangian density** Λ (excluding surface terms) is defined by

$$\Lambda[\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}] \;\; \equiv \;\; \tfrac{1}{2}\, \rho \, \langle \, \boldsymbol{v}, \boldsymbol{v} \, \rangle - \rho \, \epsilon(\rho, s) - \rho \, \mathrm{D}_t \phi - \rho s \, \mathrm{D}_t \psi + \langle \boldsymbol{A}, \, \overline{\boldsymbol{L}}_{\partial_\tau} \boldsymbol{\omega} \, \rangle.$$

The action principle is $\qquad \delta\, I = \int \delta\, \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A}) \; \mathrm{d}^4 x = 0.$

In order to derive the Euler-Lagrange equation, we consider infinitesimal transformation:

$$\begin{aligned} \boldsymbol{x} \quad &\to \quad \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t), \\ \mathrm{d}^3\boldsymbol{x} \quad &\to \quad \mathrm{d}^3\boldsymbol{x}' = (1 + \partial_k \xi^k) \mathrm{d}^3\boldsymbol{x}, \end{aligned}$$

Variations :

$$\begin{aligned} \Delta(\mathrm{d}^3\boldsymbol{x}) \quad &= \quad \partial_k \xi^k \, \mathrm{d}^3\boldsymbol{x}, \\ \Delta\rho \quad &= \quad -\rho \; \partial_k \xi^k, \qquad \Delta\boldsymbol{v} = \mathrm{D}_t \boldsymbol{\xi}, \qquad \Delta s = 0 \\ \Delta I \quad &= \quad \int \mathrm{d}^4 x \, \Big[\frac{\partial\Lambda}{\partial\boldsymbol{v}} \Delta\boldsymbol{v} + \frac{\partial\Lambda}{\partial\rho} \, \Delta\rho + \frac{\partial\Lambda}{\partial s} \, \Delta s + \Lambda \, \partial_k \xi^k \Big]. \end{aligned}$$

Substituting the expressions for $\Delta\rho$, $\Delta\boldsymbol{v}$, Δs and $\Delta(\mathrm{d}^3\boldsymbol{x})$, and requiring ΔI vanish for arbitrary variation ξ^k, we obtain the **Euler-Lagrange equation**:

$$\frac{\partial}{\partial t}\Big(\frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^l}\Big(v^l \, \frac{\partial\Lambda}{\partial v^k}\Big) + \frac{\partial}{\partial x^k}\Big(\Lambda - \rho \, \frac{\partial\Lambda}{\partial\rho}\Big) = 0. \qquad (9)$$

The Euler-Lagrange equation reduces to the momentum conservation:

$$\partial_t\big(\rho\boldsymbol{v}\big) + \nabla\cdot\rho\boldsymbol{v}\boldsymbol{v} + \nabla\, p = 0. \tag{10}$$

This becomes the Euler's equation of motion by (12):

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\,\nabla\, p. \tag{11}$$

From the invariance of I with respect to the variations of ϕ, ψ and $\boldsymbol{A}$,

$$\begin{array}{rclr} \Delta\phi & : & \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0 & \text{(Continuity eq.)}, \\ \Delta\psi & : & \partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) = 0 & \text{(Entropy eq.)}, \\ \Delta\boldsymbol{A} & : & \partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) = 0 & \text{(Vorticity eq.)}. \end{array} \tag{12}$$

The energy equation (5) can be transformed to the following equation of energy conservation:

$$\partial_t\left[\rho(\tfrac{1}{2}\,v^2 + \epsilon)\right] + \partial_k\left[\rho v^k\,(\tfrac{1}{2}\,v^2 + h)\right] = 0. \tag{13}$$

Transformation between $\boldsymbol{a}$ and $\boldsymbol{x}(\boldsymbol{a})$

- Transformation from the $\boldsymbol{a}$ space to the $\boldsymbol{x}$ space is determined by nine components of the matrix $\partial x^k/\partial a^l$.
- Remaining three conditions are given by the following two-form equation Ω^2 connecting $\boldsymbol{\omega}$ in the $\boldsymbol{x}$-space and $\boldsymbol{\Omega}_a$ in the $\boldsymbol{a}$-space:

$$\begin{aligned}\Omega^2 &= \Omega_a\,\mathrm{d}b\wedge\mathrm{d}c + \Omega_b\,\mathrm{d}c\wedge\mathrm{d}a + \Omega_c\,\mathrm{d}a\wedge\mathrm{d}b \\ &= \omega_x\,\mathrm{d}y\wedge\mathrm{d}z + \omega_y\,\mathrm{d}z\wedge\mathrm{d}x + \omega_z\,\mathrm{d}x\wedge\mathrm{d}y,\end{aligned}$$

  where $\boldsymbol{\Omega}_a = \nabla_a \times \boldsymbol{V}_a = (\Omega_a, \Omega_b, \Omega_c)$ is the vorticity in the $\boldsymbol{a}$-space, and $\boldsymbol{\omega} = \nabla\times\boldsymbol{v} = (\omega_x, \omega_y, \omega_z)$ is the vorticity in the $\boldsymbol{x}$-space.
  For example, Ω_a is given by

$$\Omega_a = \omega_x\,(\partial_b y\,\partial_c z - \partial_c y\,\partial_b z) + \omega_y\,(\partial_b z\,\partial_c x - \partial_c z\,\partial_b x) + \omega_z\,(\partial_b x\,\partial_c y - \partial_c x\,\partial_b y),$$

  with Ω_b and Ω_c being determined cyclically.

- Transformation relations of the three vectors $\boldsymbol{v}$, $\mathcal{A}$ and $\boldsymbol{\omega}$ suffice to determine the nine matrix elements $\partial x^k/\partial a^l$ locally.
  - * Thus, the transformation between the Lagrangian space and Eulerian space is determined uniquely.
  - * In this sense, the equation of vorticity is essential for the uniqueness of the transformation between both spaces.

Part II. Fluid Maxwell Equations

1. **Analogy of equations** :

   Fluid mechanics and Electromagnetism

2. **Fluid Maxwell equations**

3. **Equation of sound wave**

4. **Equation of motion of a test particle**

   Colioros force $\Leftrightarrow$ Lorentz force.

5. **Uniqueness**

1. Analogy of equations of both systems

(a) Fluid mechanics

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h &= 0, & & (14)\\ \partial_t h + \boldsymbol{v}\cdot\nabla h + a^2 \nabla\cdot\boldsymbol{v} &= 0, & & (15)\\ \partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0 & \big(\text{Vorticity equation}\big). & (16) \end{aligned}$$

$$\text{Euler's equation of motion} \ : \ \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p = -\nabla h.$$

$$\text{Continuity equation} \ : \ \partial_t \rho + (\boldsymbol{v}\cdot\nabla)\rho + \rho\nabla\cdot\boldsymbol{v} = 0\,.$$

When the fluid entropy s is uniform, the enthalpy h becomes $h = h(\rho)$:

$$\begin{aligned} \frac{1}{\rho}\nabla p &= \nabla h, \quad \text{since } \ \mathrm{d}h = \frac{1}{\rho}\mathrm{d}p + T\mathrm{d}s = \frac{1}{\rho}\,\mathrm{d}p(\rho), \quad (\mathrm{d}s = 0),\\ \mathrm{d}\rho &= (\rho/a^2)\,\mathrm{d}\,h, \quad \text{since } \ \mathrm{d}p = \big(\partial p/\partial\rho\big)_{s=\mathrm{const}}\mathrm{d}\rho = a^2\,\mathrm{d}\rho,\\ & \qquad\qquad a = \sqrt{(\partial p/\partial\rho)_s} = \sqrt{\gamma p/\rho} \ : \ \text{sound speed},\\ \partial_t\rho &= (\rho/a^2)\partial_t h, \qquad \nabla\rho = (\rho/a^2)\nabla h. \end{aligned}$$

Linearization:

(Continuity eq.) (Eq.of motion)

$$\partial_t h + a_0^2 \,\mathrm{div}\, \boldsymbol{v} = 0, \qquad \partial_t \boldsymbol{v} + \nabla h = 0,$$

$$\boxed{(\partial_t^2 - a_0^2 \nabla^2)\,(h, \quad \boldsymbol{v}) = 0, \qquad \partial_t h + a_0 \,\mathrm{div}\,(a_0 \boldsymbol{v}) = 0.}$$

(b) Electromagnetism

$$\nabla \times \boldsymbol{E}^{\mathrm{em}} + \partial_t \boldsymbol{H}^{\mathrm{em}} = 0, \qquad \nabla \cdot \boldsymbol{H}^{\mathrm{em}} = 0,$$

$$\nabla \times \boldsymbol{H}^{\mathrm{em}} - \partial_\tau \boldsymbol{E}^{\mathrm{em}} = \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{E}^{\mathrm{em}} = q.$$

$\boldsymbol{E}^{\mathrm{em}}$: electric field, $\boldsymbol{H}^{\mathrm{em}}$: magnetic field, $\tau = c\,t$ (c: light speed),

The equations in vacuum are satisfied by $\boldsymbol{E}^{\mathrm{em}}$ and $\boldsymbol{H}^{\mathrm{em}}$ defined in terms of a vector potential $\boldsymbol{A}$ and a scalar potential $\phi^{(\mathrm{e})}$:

$$\boldsymbol{E}^{\mathrm{em}} = -\partial_\tau \boldsymbol{A} - \nabla \phi^{(\mathrm{e})}, \qquad \boldsymbol{H}^{\mathrm{em}} = \nabla \times \boldsymbol{A},$$

$$\boxed{(\partial_t^2 - c^2 \nabla^2)\,(\phi^{(\mathrm{e})}, \ \boldsymbol{A}) = 0, \qquad \partial_t \phi^{(\mathrm{e})} + c \,\mathrm{div}\, \boldsymbol{A} = 0.}$$

There is analogy in the wave property.

2. Fluid Maxwell Equations

Definition: $$\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla \times \boldsymbol{v}, \tag{17}$$

$$\begin{array}{llll} \text{(A)} & \nabla \cdot \boldsymbol{H} = 0, & \text{(B)} & \nabla \times \boldsymbol{E} + \partial_t \boldsymbol{H} = 0, \\ \text{(C)} & \nabla \cdot \boldsymbol{E} = q, & \text{(D)} & a_0^2 \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{J}. \\ & q = \nabla \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}], & & \boldsymbol{J} = \partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}). \end{array}$$

From $\partial_t(\mathrm{C}) + \mathrm{div}(\mathrm{D})$, we have the *charge conservation*:

$$\partial_t q + \mathrm{div} \boldsymbol{J} = 0.$$

From the **equation (14)**, we obtain

$$\boldsymbol{E} \ (\equiv -\partial_t \boldsymbol{v} - \nabla h) \ = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \quad = \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\tfrac{1}{2} v^2). \tag{18}$$

Therefore, the *charge density* q is given by $q = \nabla \cdot \boldsymbol{E} = \mathrm{div}\big[(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\big]$.
Equation of motion is

$$\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v} + \nabla(\tfrac{1}{2} v^2 + h) = 0. \tag{14}$$

(A) is deduced from the definition of $\boldsymbol{H} = \nabla \times \boldsymbol{v}$.

(B) is just curl [Eq.(14)]. **($\Rightarrow$ Vorticity equation).**

(C) is just div [Eq.(18)].

(D) is what ?

- Eq.(D) can be derived from the **continuity equation**:

$$\partial_t h + (\boldsymbol{v} \cdot \nabla) h + a^2 \nabla \cdot \boldsymbol{v} = 0.$$

Applying ∂_t to $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla\, h$, and substituting the above,

$$-\partial_t \boldsymbol{E} - \partial_t^2 \boldsymbol{v} = \nabla \partial_t h = -\nabla \Big(a^2 \nabla \cdot \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) h \Big).$$

This can be rewritten in the form of Eq.(D):

$$a_0^2\, \nabla \times \boldsymbol{H} - \partial_t \boldsymbol{E} = \boldsymbol{J},$$

where $\quad \boldsymbol{J} = \partial_t^2 \boldsymbol{v} + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}) - \nabla \big(a^2\, \nabla \cdot \boldsymbol{v} \big) - \nabla \big((\boldsymbol{v} \cdot \nabla) h \big).$

Using the identity,

$$\nabla(\nabla \cdot \boldsymbol{v}) = \nabla \times (\nabla \times \boldsymbol{v}) + \nabla^2 \boldsymbol{v},$$

the vector $\boldsymbol{J}$ can be rewritten as

$$\boldsymbol{J} = (\partial_t^2 - a^2 \nabla^2) \boldsymbol{v} + (a_0^2 - a^2) \nabla \times (\nabla \times \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}) \nabla a^2 - \nabla (\boldsymbol{v} \cdot \nabla\, h).$$

3. Equation of Sound Wave

Sound wave is analogous to the Electromagnetic wave

Assume that a flow is generated in a uniform state at rest,
where the pressure is p_0, density ρ_0 and enthalpy h_0.

Differentiating Eq.(D) with respect to t, and eliminating $\partial_t \boldsymbol{H}$ by (B), we obtain $\partial_t^2 \boldsymbol{E} + a_0^2 \nabla \times (\nabla \times \boldsymbol{E}) = -\partial_t \boldsymbol{J}$. This reduces to

$$\begin{aligned}
\mathrm{grad}\Big[(a_0^{-2}\,\partial_t{}^2 - \nabla^2)\,h \quad &- \quad \nabla \cdot \boldsymbol{E} - \partial_t Q\Big] = 0, \qquad (19) \\
\nabla \cdot \boldsymbol{E} \quad &= \quad \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}) + \nabla^2 \tfrac{1}{2}\, v^2, \\
Q \quad &= \quad (1 - (a/a_0)^2)\nabla \cdot \boldsymbol{v} - a_0{}^{-2} (\boldsymbol{v} \cdot \nabla) h.
\end{aligned}$$

Integrating (19), we obtain the following **wave equation**:

$$(a_0^{-2}\,\partial_t{}^2 - \nabla^2)\,\tilde{h} \;=\; S(\boldsymbol{x}, t), \qquad S \equiv \nabla \cdot \boldsymbol{E} + \partial_t Q,$$

where $S(\boldsymbol{x}, t)$ is a source term of the wave.

- Since $\nabla \cdot \boldsymbol{E} = \mathrm{div}(\boldsymbol{\omega} \times \boldsymbol{v}) + \nabla^2 \frac{1}{2}\, v^2$, motion of $\boldsymbol{\omega}$ can generate waves.
  $\Rightarrow$ **Vortex sound**.
- The second term $\partial_t Q$ is $O(M^2)$.
  $\Rightarrow$ Namely, higher order if Mach number $M = |\boldsymbol{v}|/a_0$ is small.

4. Equation of motion of a Test Particle in a flow field

This is another example of analogy.

Suppose that a *test particle* of mass m is placed in a flow field $\boldsymbol{v}(\boldsymbol{x}, t)$, which is **unsteady, rotational and compressible**.

- The particle is assumed to be sufficiently small, so that its influence is regarded as perturbation and the background velocity field is $\boldsymbol{v}(\boldsymbol{x}, t)$ is independent of the position and velocity of the particle.

- Equation of motion of the particle can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P} = m\,\boldsymbol{E} + m\,\boldsymbol{u}\times\boldsymbol{H} - m\,\nabla\phi_g, \qquad \boldsymbol{P} = (P_i). \tag{E}$$

$$P_i = mu_i + m_{ik}u_k, \qquad \boldsymbol{E} = -\partial_t\boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla\times\boldsymbol{v}.$$

- P_i : Total *additional momentum* due to the particle existence.
- $\boldsymbol{u} = (u_i)$: Particle velocity *relative to* the fluid velocity $\boldsymbol{v}$.
- m_{ik} : *Induced mass tensor* of the particle.
- $\frac{1}{2}m_{ik}u_iu_k$: *Induced kinetic energy* due to the particle.

Particle position is expressed by $\boldsymbol{x}_p(t) = \boldsymbol{\xi}(t) + \boldsymbol{X}(t)$.
Total particle velocity is $\boldsymbol{u}+\boldsymbol{v}$, where

$$\boldsymbol{u}(t) = \mathrm{d}\boldsymbol{\xi}/\mathrm{d}t, \qquad \boldsymbol{v}(t, \boldsymbol{x}_p(t)) = \mathrm{d}\boldsymbol{X}/\mathrm{d}t.$$

- The equation (E) can be derived from the following Lagrangian :

$$L(t, \boldsymbol{\xi}, \boldsymbol{u}) = \tfrac{1}{2}\, m\, (\boldsymbol{u} + \boldsymbol{v})^2 + \tfrac{1}{2}\, m_{jk} u_j u_k - m\, \phi, \tag{20}$$

  where $m\phi$ is a force potential. Lagrange's equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{\xi}_i}\Big) = \frac{\partial L}{\partial \xi_i}, \qquad \text{where} \quad \frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \boldsymbol{u} \cdot \nabla.$$

  From this, we obtain the equation (E):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m u_i) + \frac{\mathrm{d}}{\mathrm{d}t}(m_{ik} u_k) &= -m\, \partial_t \boldsymbol{v} + m \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) - m \nabla \phi. \qquad (21) \\ m \nabla \phi &= m\, \phi_g + m\, h, \qquad P_i = m u_i + m_{ik} u_k, \end{aligned}$$

  Rewriting, $$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{P} = m\, \boldsymbol{E} + (m/a)\, \boldsymbol{u} \times a \boldsymbol{H} - m\, \nabla \phi_g. \tag{E}$$

- The equation of motion of a charged particle in an electromagnetic field is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m \boldsymbol{v}_p) = e\, \boldsymbol{E}^{\mathrm{em}} + (e/c)\, \boldsymbol{v}_p \times \boldsymbol{H}^{\mathrm{em}} - m \nabla \Phi_g. \tag{F}$$

  $\boldsymbol{v}_p$: velocity of a charged particle, c : the light velocity.

  $\boldsymbol{H}^{\mathrm{em}} \Leftrightarrow a\, \boldsymbol{H}$.

- Equivalence of both expressions reconfirms validity of the definitions

  $\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \qquad \boldsymbol{H} = \nabla \times \boldsymbol{v}.$

5. Uniqueness

We consider here whether the original system of fluid equations (14), (15) and (16) are recovered from the modified fluid Maxwell equations:

$$
\begin{array}{llll}
\text{(A)} & \nabla\cdot\boldsymbol{H}=0, & \text{(B)} & \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v})+\partial_t\boldsymbol{H}=0,\\
\text{(C)} & \nabla\cdot\boldsymbol{E}=q, & \text{(D)} & a_0^2\,\nabla\times\boldsymbol{H}-\partial_t\boldsymbol{E}=\boldsymbol{J}.
\end{array}
$$

$$q=\nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}],$$

$$\boldsymbol{J}=\partial_t^2\boldsymbol{v}-\nabla[\boldsymbol{v}\cdot\nabla h+a^2\nabla\cdot\boldsymbol{v}]+a_0^2\nabla\times(\nabla\times\boldsymbol{v}).$$

modified : $\nabla\times\boldsymbol{E}$ is replaced by $\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v})$.

$$
\begin{aligned}
\partial_t\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla h &= 0, && (14)\\
\partial_t h+\boldsymbol{v}\cdot\nabla h+a^2\nabla\cdot\boldsymbol{v} &= 0, && (15)\\
\partial_t\boldsymbol{\omega}+\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0. && (16)
\end{aligned}
$$

Starting assumption is that the forces acting on a test particle of mass m in flow field are represented in the same form as those of electromagnetism:

$$\mathrm{d}\boldsymbol{P}/\mathrm{d}t=m\boldsymbol{E}+m\boldsymbol{u}\times\boldsymbol{H}-m\nabla\phi_g. \tag{22}$$

We use this as a guiding principle. Comparing this with the equation (21) derived from the Lagrangian (20), we obtain the following definition of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\boldsymbol{E}=-\partial_t\boldsymbol{v}-\nabla h,\qquad \boldsymbol{H}=\nabla\times\boldsymbol{v}. \tag{23}$$

So that, Eq.(B) is equivalent to the vorticity equation.

- By the definition $\boldsymbol{H} = \nabla \times \boldsymbol{v}$, Eq.(A) is satisfied identically, and Eq.(B) is the vorticity equation (16) .
- Next is to consider the remaining equations (C) and (D), and try to deduce (14) and (15).

Let us consider Eq.(D), from which we try to derive the equation (15) under the condition of no mass source.

In fact, substituting the expressions of $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{J}$,

$$\begin{aligned}\partial_t^2 \boldsymbol{v} + \nabla \partial_t h + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}) \ &= \\ &\partial_t^2 \boldsymbol{v} - \nabla[\boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}] + a_0^2 \nabla \times (\nabla \times \boldsymbol{v}).\end{aligned}$$

From this, we obtain $\nabla\big(\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v}\big) = 0$. Therefore,

$$\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v} = C(t) \quad \text{(a function of time } t \text{ only).}$$

The term $C(t)$ must vanish in the absence of mass source. Namely, if $C(t) \neq 0$, there is a mass source of term $(\rho/a^2)C$. Thus, $C(t) = 0$, and the equation (15) is recovered:

$$\partial_t h + \boldsymbol{v} \cdot \nabla h + a^2 \nabla \cdot \boldsymbol{v} = 0.$$

Next, we consider Eq.(C). Substituting $q = \mathrm{div}[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}]$ and integrating it, we obtain

$$\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\times\xi, \qquad \xi : \textbf{arbitrary vector field.}$$

By using the definition of $\boldsymbol{E}$, this reduces to

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\, h = -\nabla\times\xi.$$

Using the identity $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(\frac{1}{2}\,v^2)$, this is rewritten as

$$\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} + \nabla(h + \tfrac{1}{2}\,v^2) = -\nabla\times\xi. \tag{C$*$}$$

On the other hand, Eq.(B) leads to

$$\partial_t\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{v} = -\nabla\varphi, \qquad \varphi : \textbf{arbitrary scalar field.} \tag{B$*$}$$

Comparing (B*) with (C*), it is found that

$$\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \qquad \nabla\times\xi = 0, \qquad \varphi = h + \tfrac{1}{2}\,v^2.$$

Thus, we obtain the **Euler's equation (14):**

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\, h = 0.$$

Thus, **the fluid equations,**

$$\boxed{\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h &= 0, \\ \partial_t h + \boldsymbol{v}\cdot\nabla h + a^2 \nabla\cdot\boldsymbol{v} &= 0, \\ \partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0. \end{aligned}}$$

are recovered

$$\Updownarrow$$

from modified **Maxwell equations:**

$$\boxed{\begin{array}{llll} \text{(A)} & \nabla\cdot\boldsymbol{H} = 0, & \text{(B)} & \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) + \partial_t\boldsymbol{H} = 0, \\ \text{(C)} & \nabla\cdot\boldsymbol{E} = q, & \text{(D)} & a_0^2\,\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = \boldsymbol{J}. \\ \multicolumn{4}{l}{q = \nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}], \quad \boldsymbol{J} = \partial_t^2\boldsymbol{v} + \nabla\partial_t h + a_0^2\nabla\times(\nabla\times\boldsymbol{v}).} \end{array}}$$

by the definition $\Big[\boldsymbol{E} = -\partial_t \boldsymbol{v} - \nabla h, \quad \boldsymbol{H} = \nabla\times\boldsymbol{v}\Big]$, and isentropy.

T. Kambe (2010): "*Geometrical Theory of Dynamical Systems and Fluid Flows*", (revised ed., World Scientific, Singapore) Chap.7.

Summary

Part I

- Lagrangian is defined by

$$\mathcal{L}^* = \int \Lambda(\boldsymbol{v}, \rho, s, \phi, \psi, \boldsymbol{A})\, \mathrm{d}^3\boldsymbol{x},$$
$$\Lambda = \tfrac{1}{2}\, \rho \,\langle\, \boldsymbol{v}, \boldsymbol{v}\,\rangle - \rho\, \epsilon(\rho, s) - \rho\, \mathrm{D}_t\phi - \rho s\, \mathrm{D}_t\psi + \langle \boldsymbol{A},\, \overline{\boldsymbol{L}}_{\partial_\tau}\boldsymbol{\omega}\,\rangle.$$

- Euler-Lagrange equation and Noether's theorem are given by

$$\text{Momentum conservation}: \quad \partial_t\big(\rho\boldsymbol{v}\big) + \nabla\cdot\rho\boldsymbol{v}\boldsymbol{v} + \nabla\, p = 0.$$
$$\text{Energy conservation}: \quad \partial_t\left[\rho(\tfrac{1}{2}\, v^2 + \epsilon)\right] + \partial_k\left[\rho v^k\,(\tfrac{1}{2}\, v^2 + h)\right] = 0.$$

Action principle with respect to the variations of ϕ, ψ and $\boldsymbol{A}$:

$$\begin{aligned}
\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) &= 0 \qquad \text{(Continuity eq.)},\\
\partial_t(\rho s) + \nabla\cdot(\rho s\boldsymbol{v}) &= 0 \qquad \text{(Entropy eq.)},\\
\partial_t\boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0 \qquad \text{(Vorticity eq.)}.
\end{aligned}$$

- Transformation relations of the three vectors $\boldsymbol{v}$, $\mathcal{A}$ and $\boldsymbol{\omega}$ suffice to determine the nine matrix elements $\partial x^k/\partial a^l$ locally.
- Thus, the vorticity equation is essential for the uniqueness of the transformation between the Lagrangian and Eulerian spaces.

Part II

- From the fluid equations,

$$
\boxed{
\begin{aligned}
\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla h &= 0,\\
\partial_t h + \boldsymbol{v}\cdot\nabla h + a^2 \nabla\cdot\boldsymbol{v} &= 0,\\
\partial_t \boldsymbol{\omega} + \nabla\times(\boldsymbol{\omega}\times\boldsymbol{v}) &= 0.
\end{aligned}
}
$$

$$\Downarrow$$

fluid Maxwell equations are derived:

$$
\boxed{
\begin{array}{llll}
\text{(A)} & \nabla\cdot\boldsymbol{H} = 0, & \text{(B)} & \nabla\times\boldsymbol{E} + \partial_t\boldsymbol{H} = 0,\\
\text{(C)} & \nabla\cdot\boldsymbol{E} = q, & \text{(D)} & a_0^2\,\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = \boldsymbol{J}.\\
\multicolumn{4}{c}{q = \nabla\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}], \quad \boldsymbol{J} = \partial_t^2\boldsymbol{v} + \nabla\partial_t h + a_0^2\nabla\times(\nabla\times\boldsymbol{v}).}
\end{array}
}
$$

by the definition $\left[\boldsymbol{E} = -\partial_t\boldsymbol{v} - \nabla h, \quad \boldsymbol{H} = \nabla\times\boldsymbol{v}\right]$, and isentropy.

- From this, equation of sound wave is derived. Hence, the sound wave is analogous to the Electromagnetic wave. Dynamical motion of vorticity can generate sound waves.

- Forces acting on a test particle of mass m in a flow field are represented in the same form as those of electromagnetism:

$$\mathrm{d}\boldsymbol{P}/\mathrm{d}t = m\boldsymbol{E} + m\boldsymbol{u}\times\boldsymbol{H} - m\nabla\phi_g.$$

Thank you very much for your attention

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